Ptolemy, the Regular Heptagon, and Quasiperiodic Tilings

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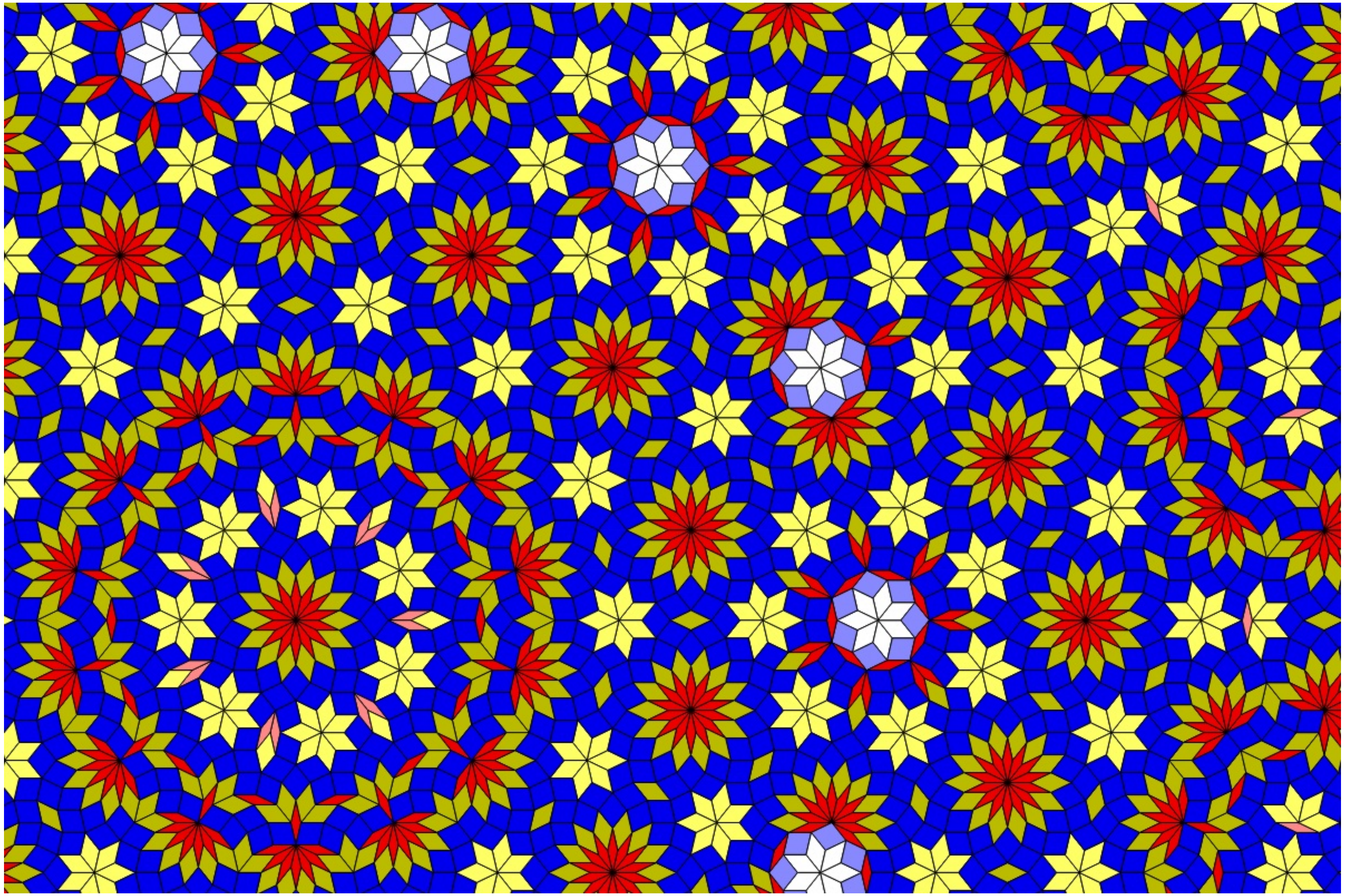
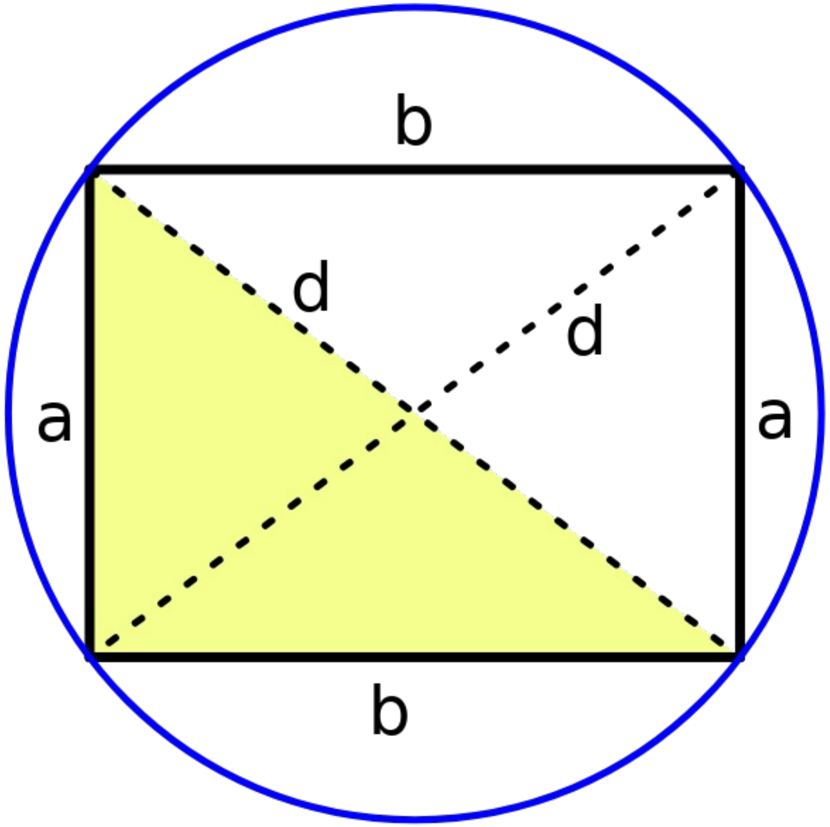
With elementary geometry and algebra you can create this:



Ptolemy's Theorem

The product of the diagonals of a **cyclic** quadrilateral is equal to the sum of the two products of opposite sides.



Example: For any rectangle

$d^2 = a^2 + b^2$

Pythagoras' theorem is a special case of Ptolemy's theorem.

Regular Pentagon

Sides of unit length.

Ptolemy's theorem:

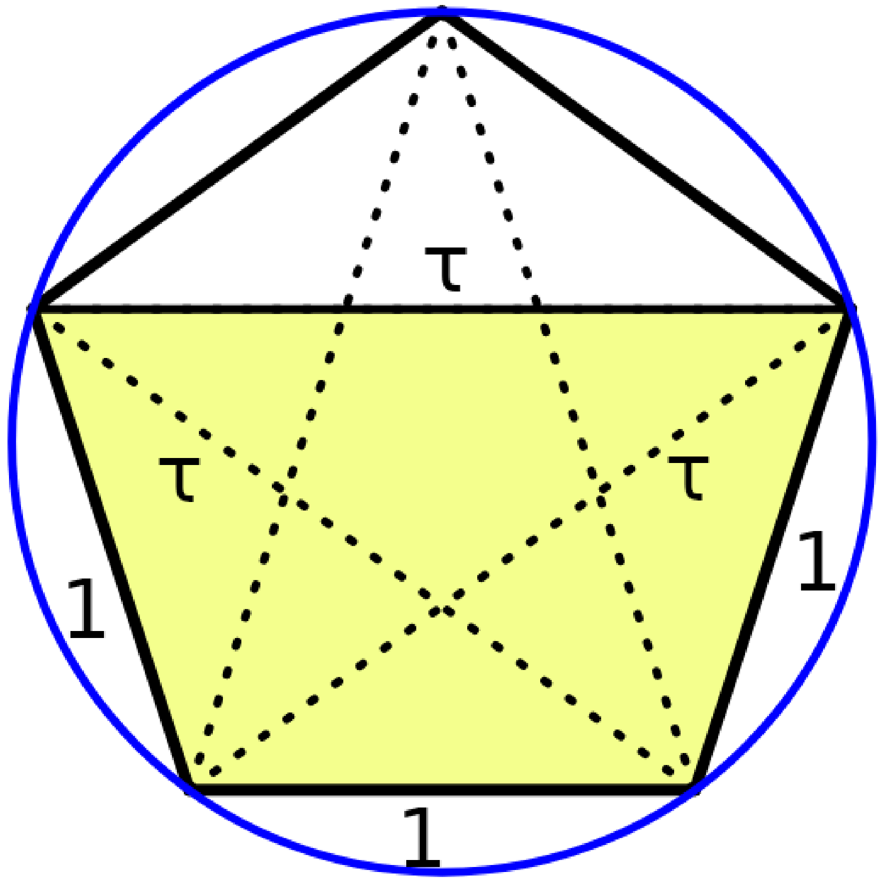
Length τ of the diagonal

$\boldsymbol{\tau^2 = \tau + 1}$

τ is the golden ratio

$\tau = (1 + \sqrt{5}) / 2$

(suggests an easy construction of the regular pentagon with ruler and compass)

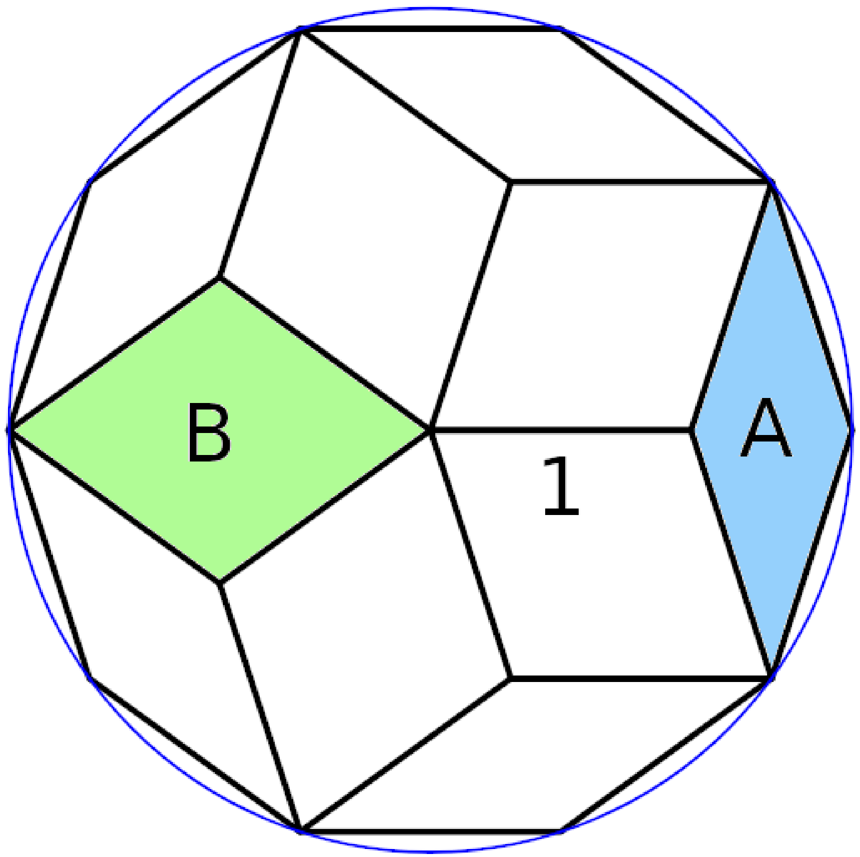


Five-fold Rosette

Rhombus A with angles $\pi/5$ and $4\pi/5$

Rhombus B with angles $2\pi/5$ and $3\pi/5$

Tiles for making quasi-periodic tilings!



Long diagonal D_B of rhombus B equals
short diagonal d_A of rhombus A plus a rhombus side:

$$D_B = 1 + d_A$$

Substitution Method

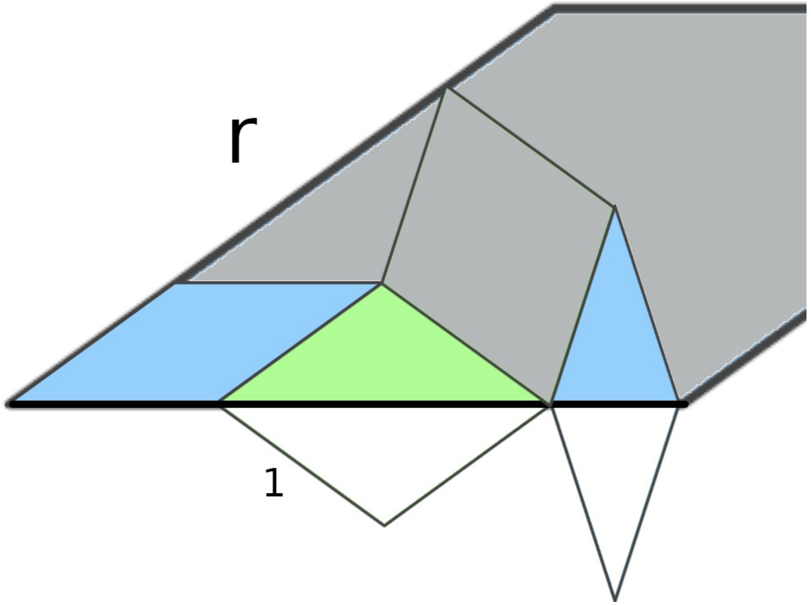
First step: Inflate tile by an inflation ratio r.

Second step: Replace inflated tiles by small tiles of original size.

Tiles have corner angles that are integer multiples of $\pi/5 = 36^\circ$ for five-fold rotational symmetry.

To get quasiperiodic tilings we have to put some rhombi with their diagonal on the border of inflated tiles.

They are cut into two triangles by the border. One of the triangles lies inside the inflated rhombus.



We can put a type B rhombus with its long diagonal D_B on the border.
It has corners with angles $2\pi/5$ on the border.
This gives triangle tiles with two corner angles of $\pi/5$.

We can a type A rhombus with its short diagonal d_A on the border.
It has corners with angles $4\pi/5$ on the border.
This gives triangle tiles with two corner angles of $2\pi/5$.

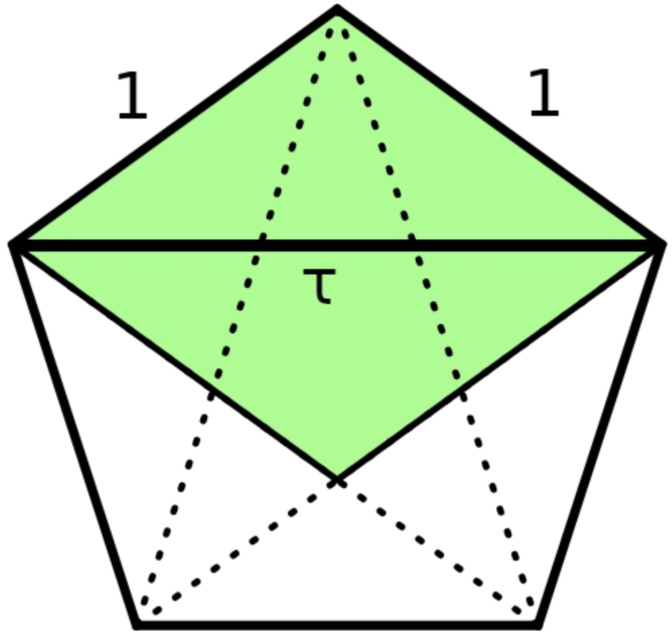
We can also put rhombi with their sides on the border.

The sum of these lengths gives the inflation ratio $r = m + n\, d_A + p\, D_B$ (integer m,n and p)

Length of Diagonals

A small type B rhombus has the same diagonal length as the pentagon.

$D_B = \tau$



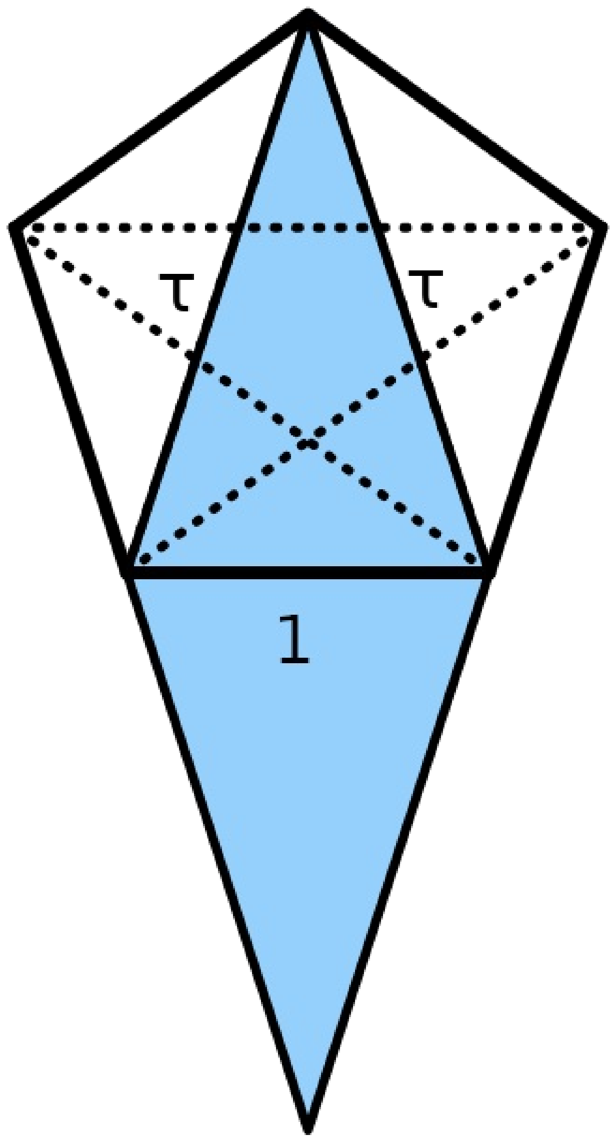
An inflated type A rhombus fits the pentagon

$d_A = 1/\tau = \tau - 1$ (sides of unit length)

because $\tau^2 = \tau + 1$

With these results the inflation ratio becomes

$r = m + n\, d_A + p\, D_B = h + k\,\tau$ (integer h and k)



Conservation of Area

The total area of the small rhombi is **exactly** equal to the area of the inflated rhombus.

For the areas A and B of the two rhombi:

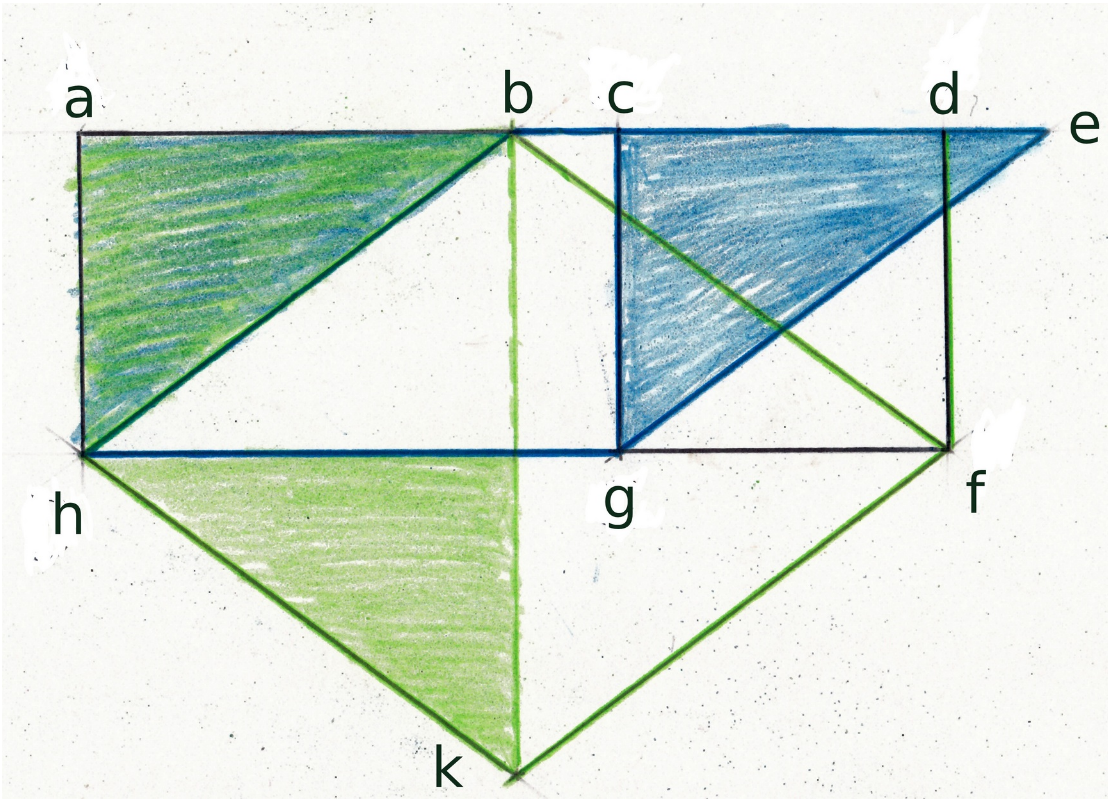
The type A rhombus (blue lines, hgeb) has the same area as the hgca rectangle.

The type B rhombus (green lines, hbfk) has the same area as the hfda rectangle.

The two rectangles have the same height.

The length hg is equal to the side of the rhombi.

The length hf is equal to the long diagonal of the B type rhombus.



Thus we get for their surfaces that B = τ A.

Substitution of Areas

Areas increase as the square of the inflation ratio:

$r^2 = (h + k\tau)^2 = h^2 + k^2 + (2hk + k^2)\tau$

Area of an inflated rhombus of type A

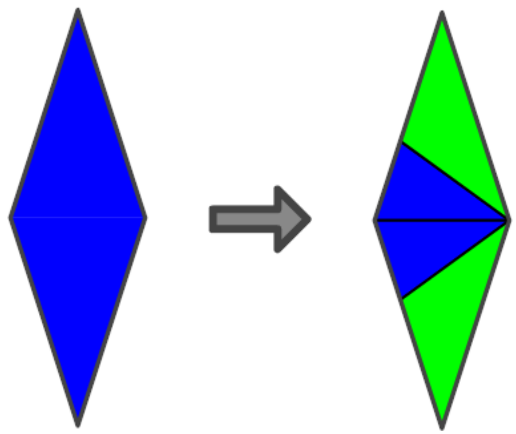
$\mathbf{A} = r^2 A = (h^2 + k^2) A + (2hk + k^2)\tau A = (h^2 + k^2) A + (2hk + k^2) B$

Note that h and k are integers, and τ is irrational.

Thus the number of small A rhombi in an inflated A rhombus is: $(h^2 + k^2)$

And the number of small B rhombi in an inflated A rhombus is: $(2hk + k^2)$

For the Penrose rhombus tiling:



Inflation ratio r = τ, with h=0, k=1

Substitution at border: $\tau = D_B = 1 + d_A$

$\mathbf{A} = A + B$

Area of an inflated B rhombus

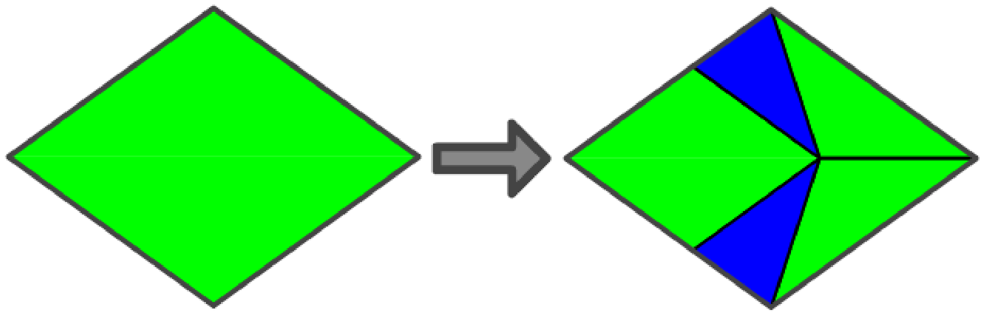
$\mathbf{B} = \tau\, \mathbf{A} = (h^2 + k^2)\, \tau A + (2hk + k^2)(\tau + 1)\, A = (2hk + k^2)\, A + (h^2 + 2hk + 2k^2)\, B$

Thus the number of small A rhombi in an inflated B rhombus is: $(2hk + k^2)$

And the number of small B rhombi in an inflated B rhombus is: $(h^2 + 2hk + 2k^2)$

For the Penrose rhombus tiling:

$\mathbf{B} = A + 2B$



Substitution at Diagonals

Consider a type A rhombus that has its short diagonal on the border of an inflated rhombus. The border of the inflated rhombus cuts it at its short diagonal into two triangles.

These triangles belong to different inflated tiles.
The substitutions for these two triangles have to match.

Thus type A rhombi should have a substitution that is mirror symmetric at the short diagonal.

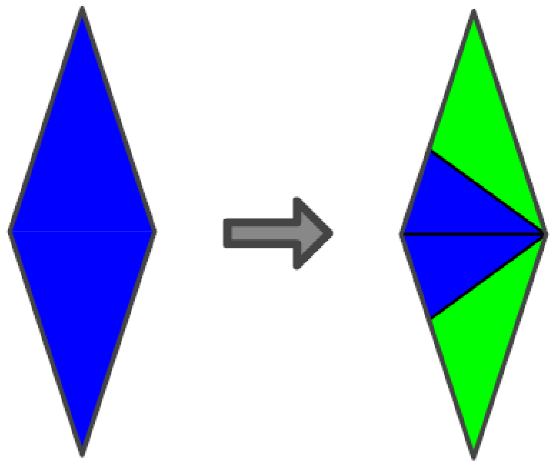
Short diagonal of an inflated type A rhombus:

$\mathbf{d}_A = r\, d_A = (h + k\,\tau)\,(\tau - 1) = k - h + h\,\tau$

This leaves only a few possibilities for the substitution at the short diagonal, similar to the substitution at the border.

For the Penrose rhombus tiling:

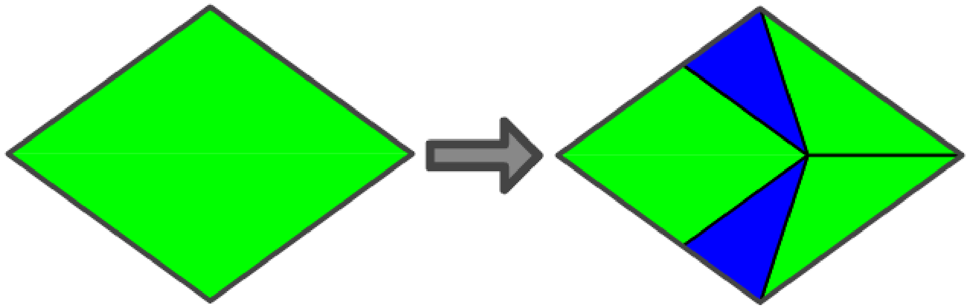
$\mathbf{d}_A = 1$



The substitution for a type B rhombus should be mirror symmetric at the long diagonal.

Long diagonal of an inflated B rhombus:

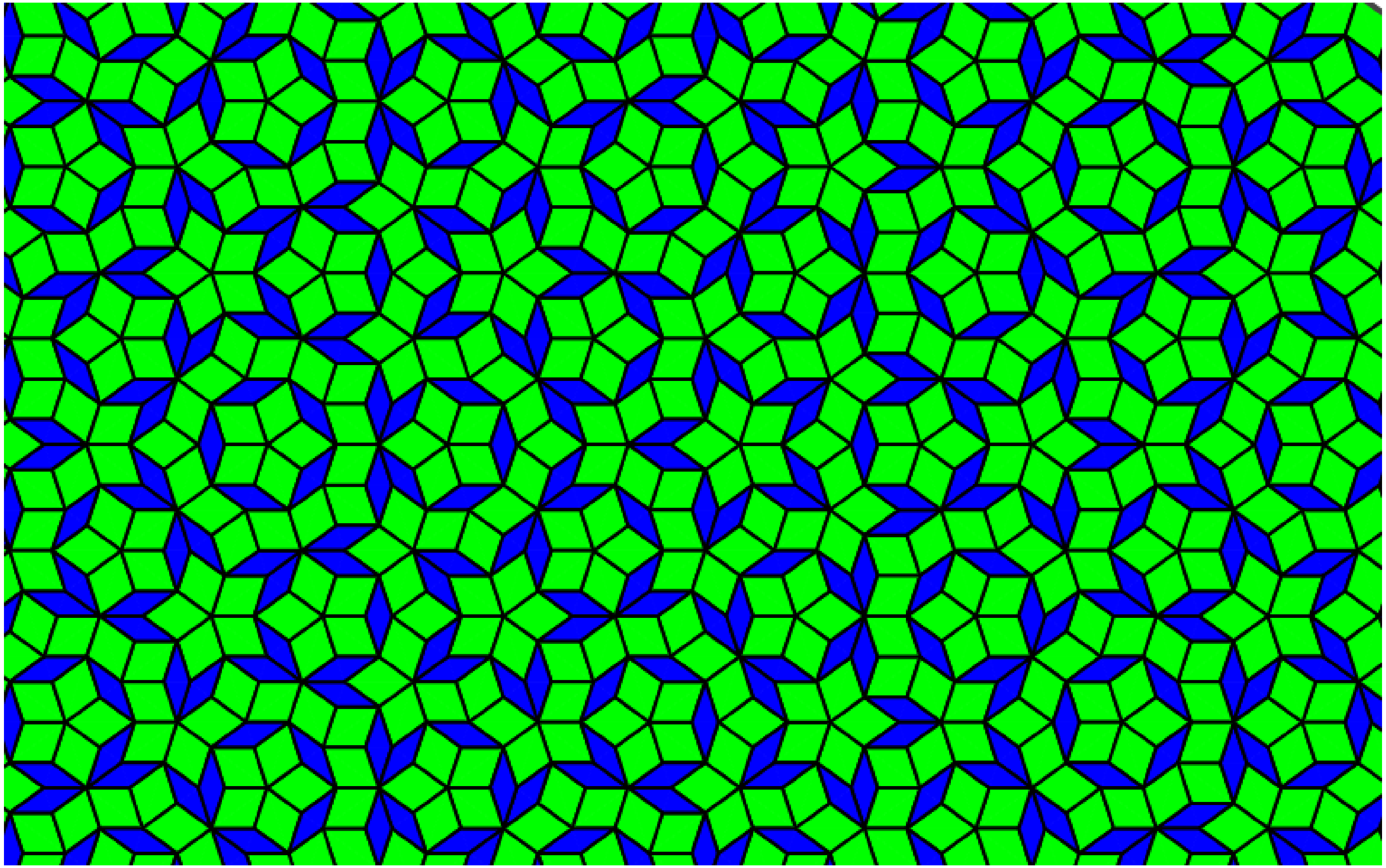
$\mathbf{D}_B = r\, D_B = (h + k\,\tau)\,\tau = k + (h + k)\,\tau$



For the Penrose rhombus tiling:

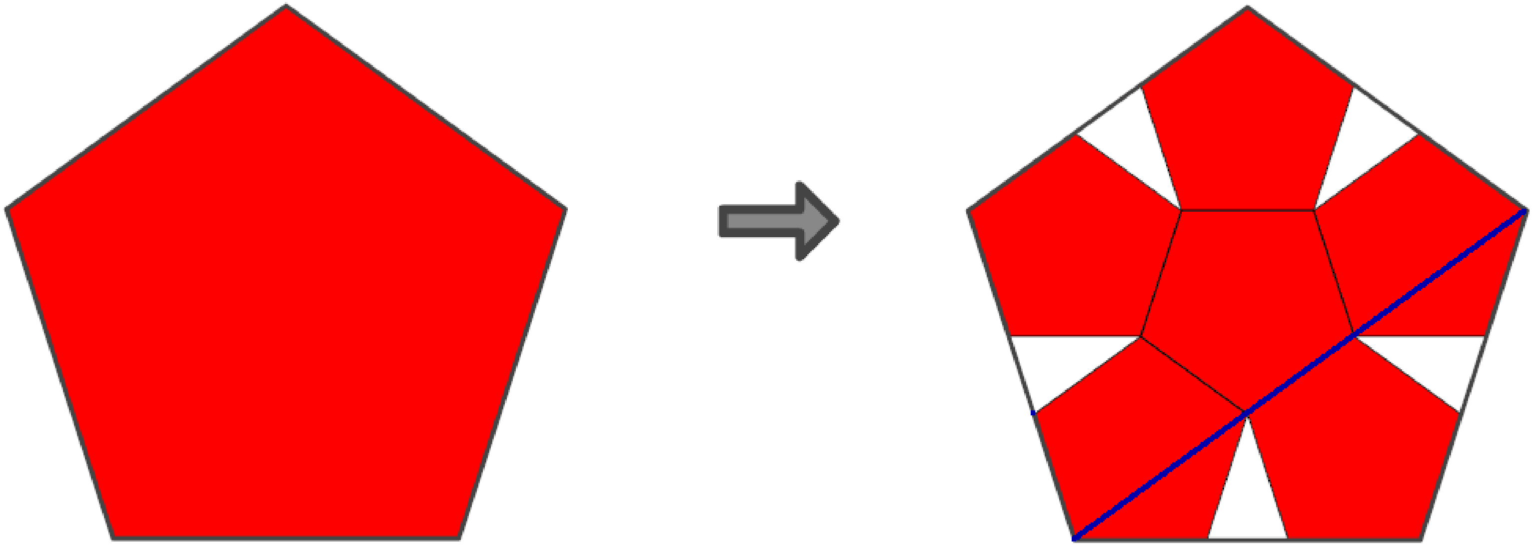
$\mathbf{D}_B = 1 + \tau = 1 + D_B$

Penrose Rhombus Tiling



Pentagonal Penrose Tiling (previous talk by Jennifer Padilla)

Substitution

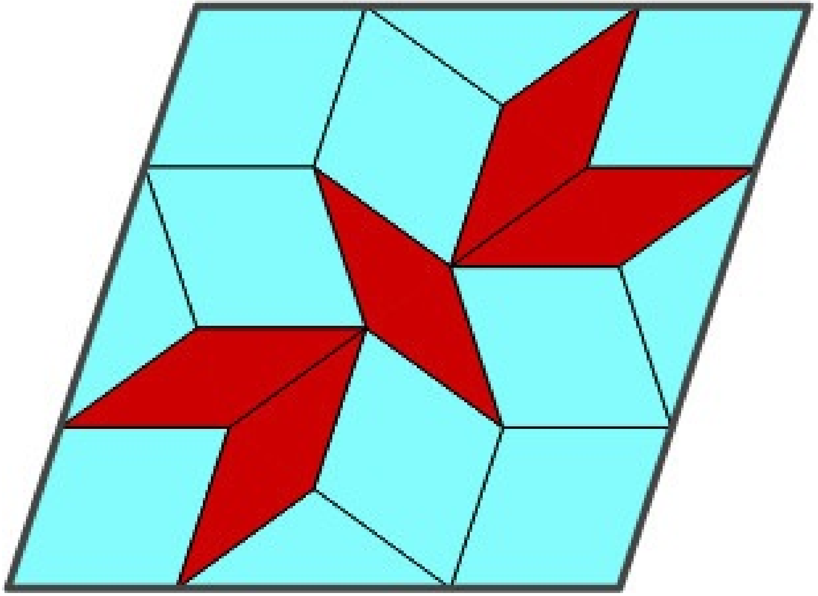


This gives for the self-similarity ratio $\tau\, r = 2\,\tau + 1$ => $r = 2 + 1/\tau = \tau + 1 = \tau^2$

Doing twice the rhombus substitution we get the same ratio.

This hides the golden ratio?

Another Tiling of Five-Fold Rotational Symmetry



Inflation ratio: $r = 2 + \tau = 1 + D_B + 1$

Symmetric substitution at border.

Mirror symmetric substitutions at **both** diagonals?

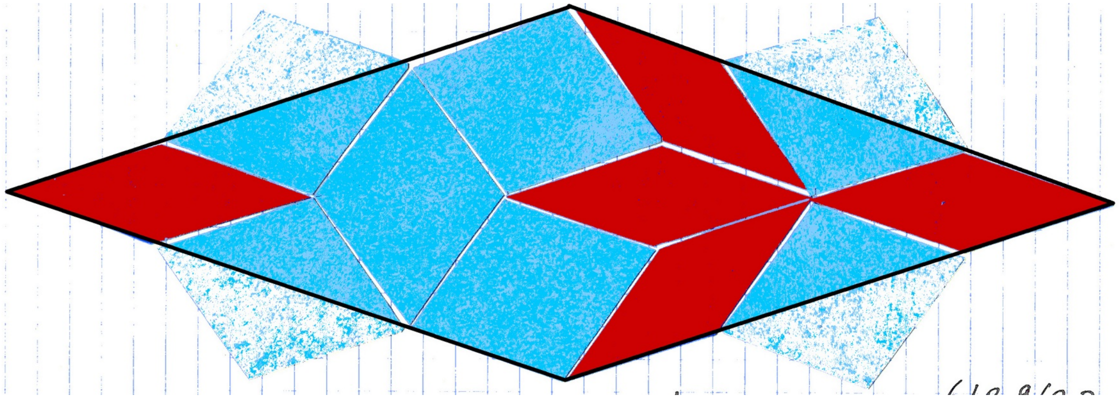
Easy to find for type B rhombus:

B = 5 A + 10 B

A type A rhombus has to be at the center!

$\mathbf{D}_B = 1 + 3\,\tau$
$\cancel{= 1 + 3\,D_B}$
$\cancel{= 3 + D_B + 2\,d_A}$
$= 2\,D_B + d_A + 2$

Problem with the Type A Rhombus



A = 5 A + 5 B

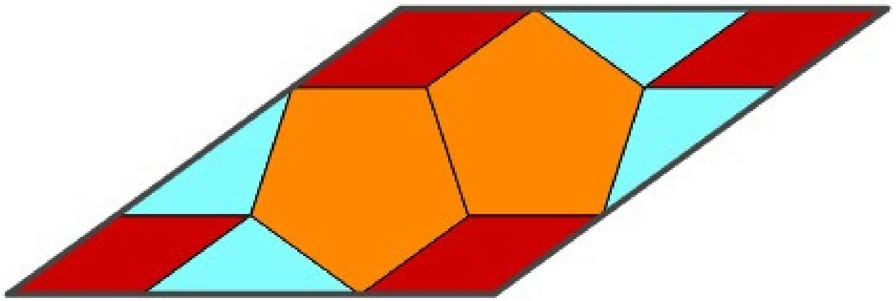
We can't put both a type A rhombus and a type B rhombus at the center.

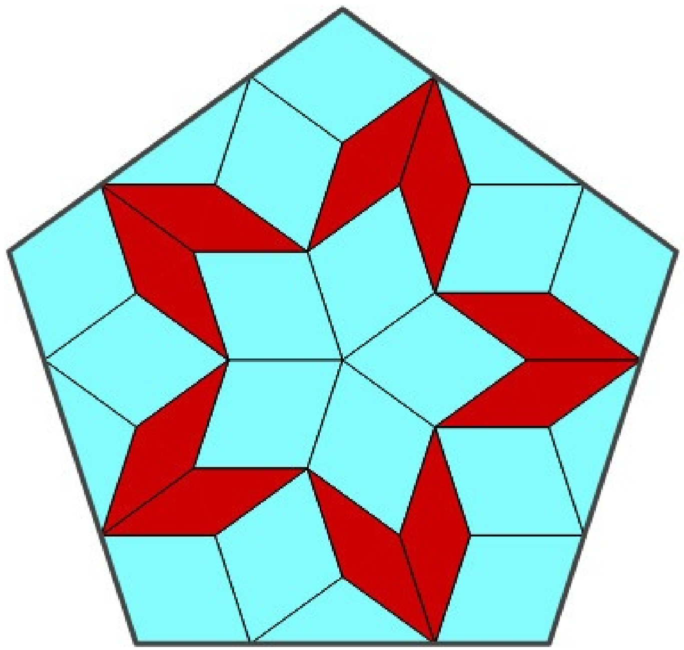
No symmetric solution at both diagonals with rhombi only.

$\mathbf{d}_A = 2\,\tau - 1$

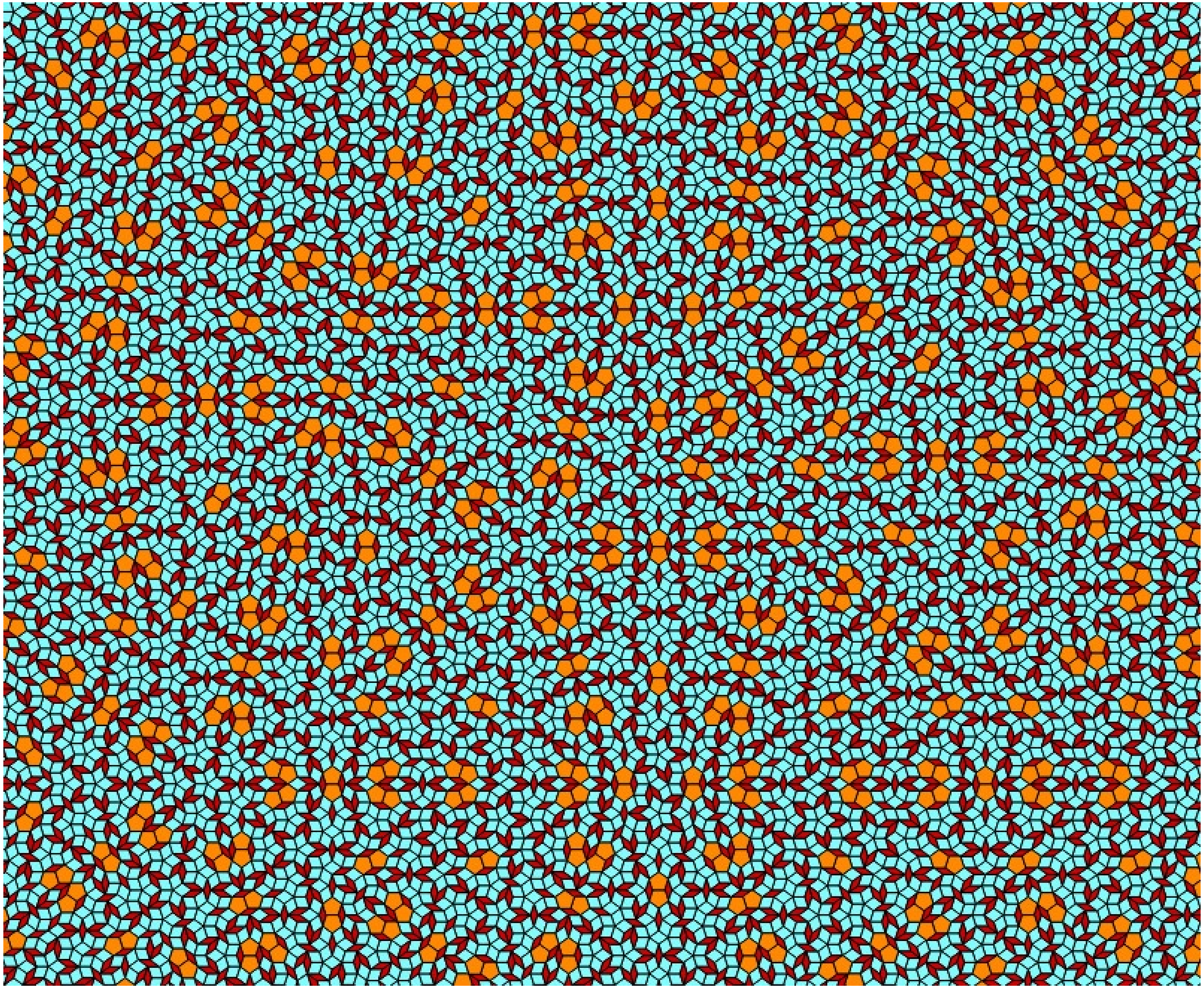
$\quad = 2\, d_A + 1$

We need regular pentagons as additional tiles



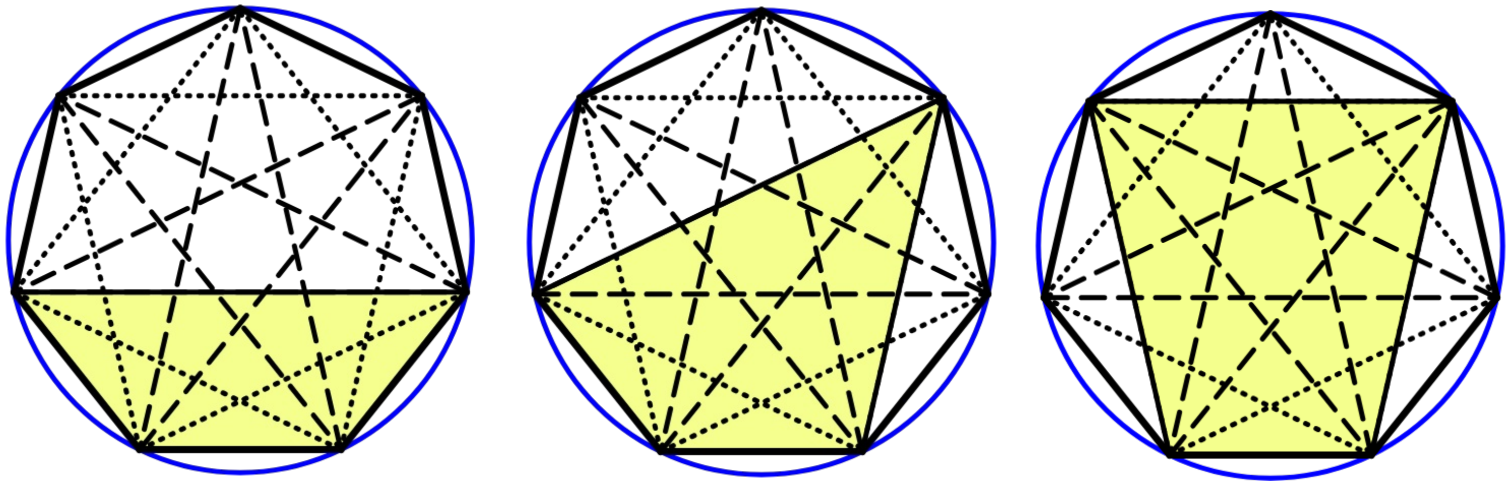


The Tiling



Ptolemy's Theorem and the Regular Heptagon

The regular heptagon has a short diagonal φ and a long diagonal ρ.



$\varphi^2 = \rho + 1$
(similar to $\tau^2 = \tau + 1$)

$\varphi \rho = \varphi + \rho$

$\rho^2 = \varphi^2 + \varphi = 1 + \rho + \varphi$

We use these results for calculating substitution rules.

Linear combinations of 1, φ, and ρ with integer coefficients define a commutative ring of algebraic integers. φ and ρ are units.

φ, ρ, and $\varphi / \rho = \varphi - 1$ are irrational numbers.

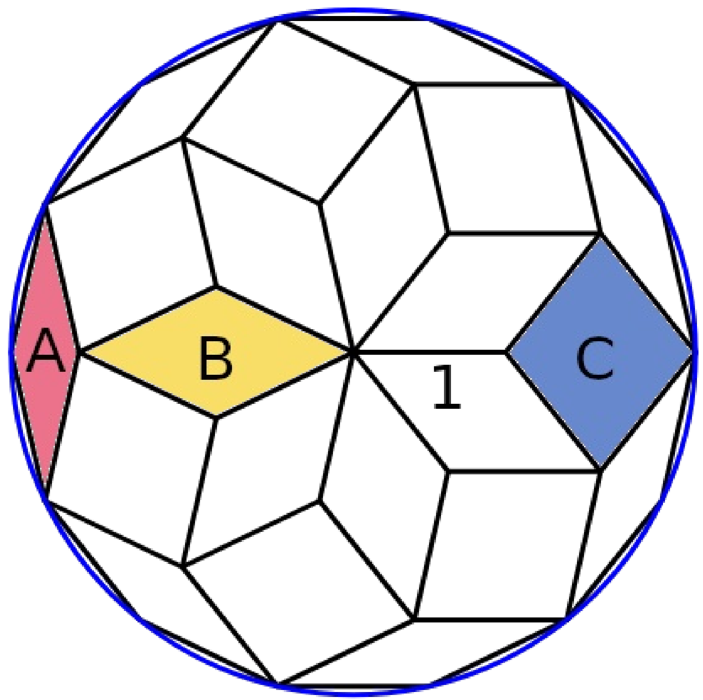
Seven-Fold Rosette

Rhombus A with angles $\pi/7$ and $6\pi/7$

Rhombus B with angles $2\pi/7$ and $5\pi/7$

Rhombus C with angles $3\pi/7$ and $4\pi/7$

Use as tiles for quasiperiodic tilings!



$d_A + D_B = d_C + 1$

Long diagonal D_B of rhombus B plus short diagonal d_A of rhombus A equal short diagonal of rhombus C plus rhombus side

Matching rhombi to the heptagon we get for their diagonals

$d_A = \rho - \varphi$

$D_B = \varphi$

$d_C = \rho - 1$

Inflation Ratios and Substitutions

We proceed as for the five-fold tilings.

Inflation ratio from the substitution at the border of inflated tiles $r = m + n\, d_A + p\, D_B + q\, d_C$

Using the results for the diagonals: $r = h + k\, \varphi + j\, \rho$

From r^2 we get the substitution of the area of inflated tiles.

For mirror symmetry we get constraints on the substitution at diagonals.

Various tilings have been found previously:

Ludwig Danzer, $r = 1 + \varphi$
Chaim Goodman-Strauss, $r = 1 + 2\, \varphi$
Joshua Socolar, $r = 1 + 2\, \varphi + \rho$
Alexey Madison, Theo Schaad, $r = 1 + \varphi + \rho$

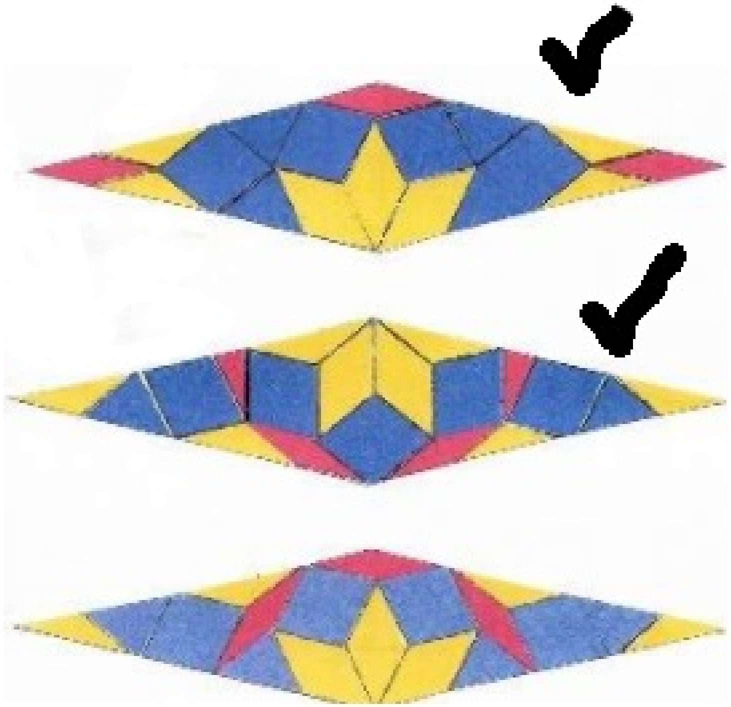
Theo's puzzle, Substitutions for Type A Rhombi

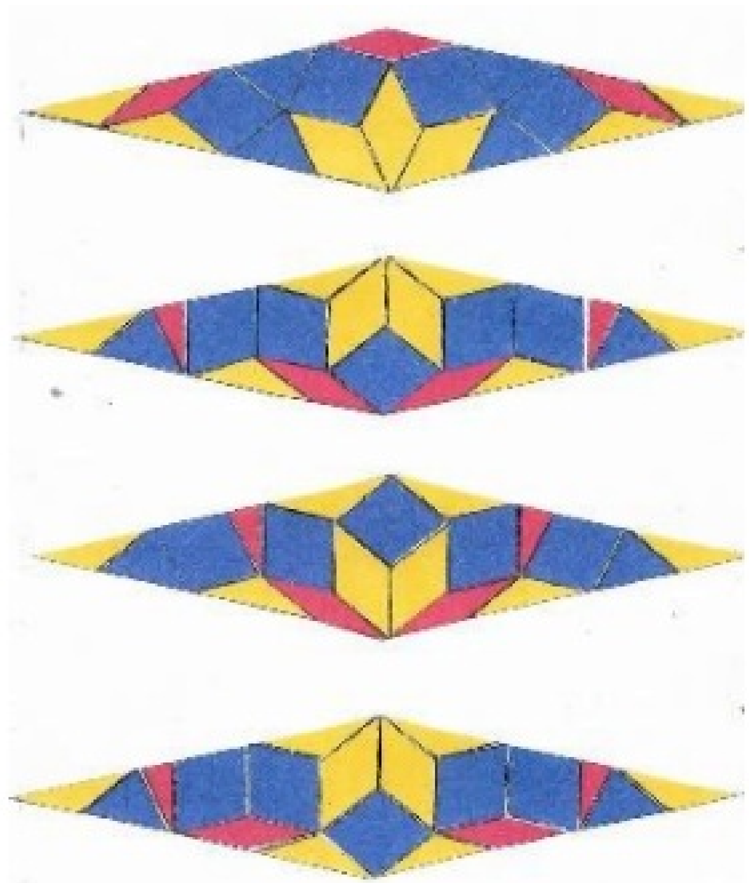
Theo used the inflation ratio $r = 1 + \varphi + \rho = 1 + d_A + 2\ D_B = 2 + D_B + d_C$

He created a collection of substitutions with mirror symmetry at a diagonal as puzzle pieces.
Then he chose among these pieces and fitted them together to get a tiling.

$\mathbf{A} = 3\ A + 5\ B + 6\ C$

$\mathbf{d}_A = d_A + D_B$
$\quad = 1 + d_C$

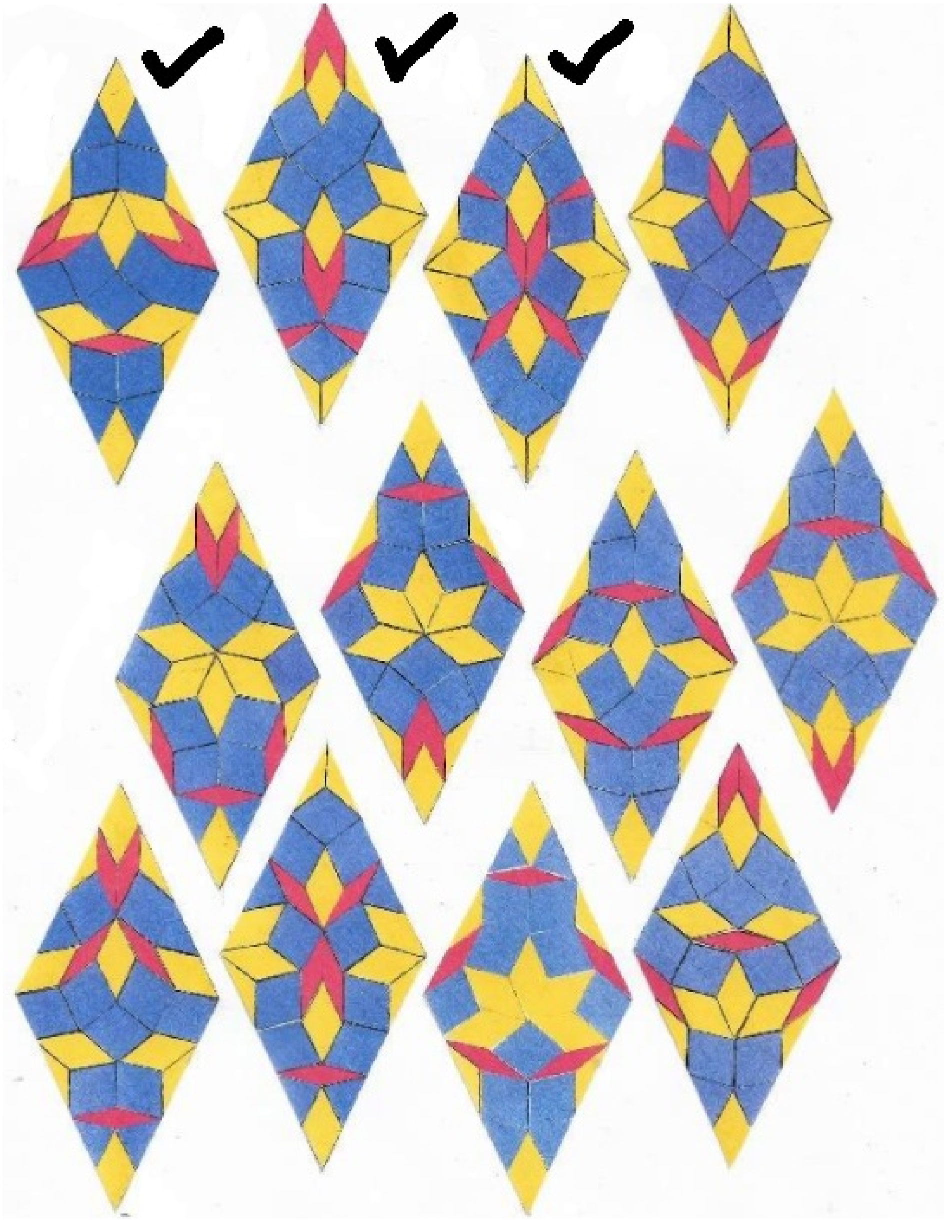




Substitutions for Type B Rhombi

B = 5 A + 9 B + 11 C

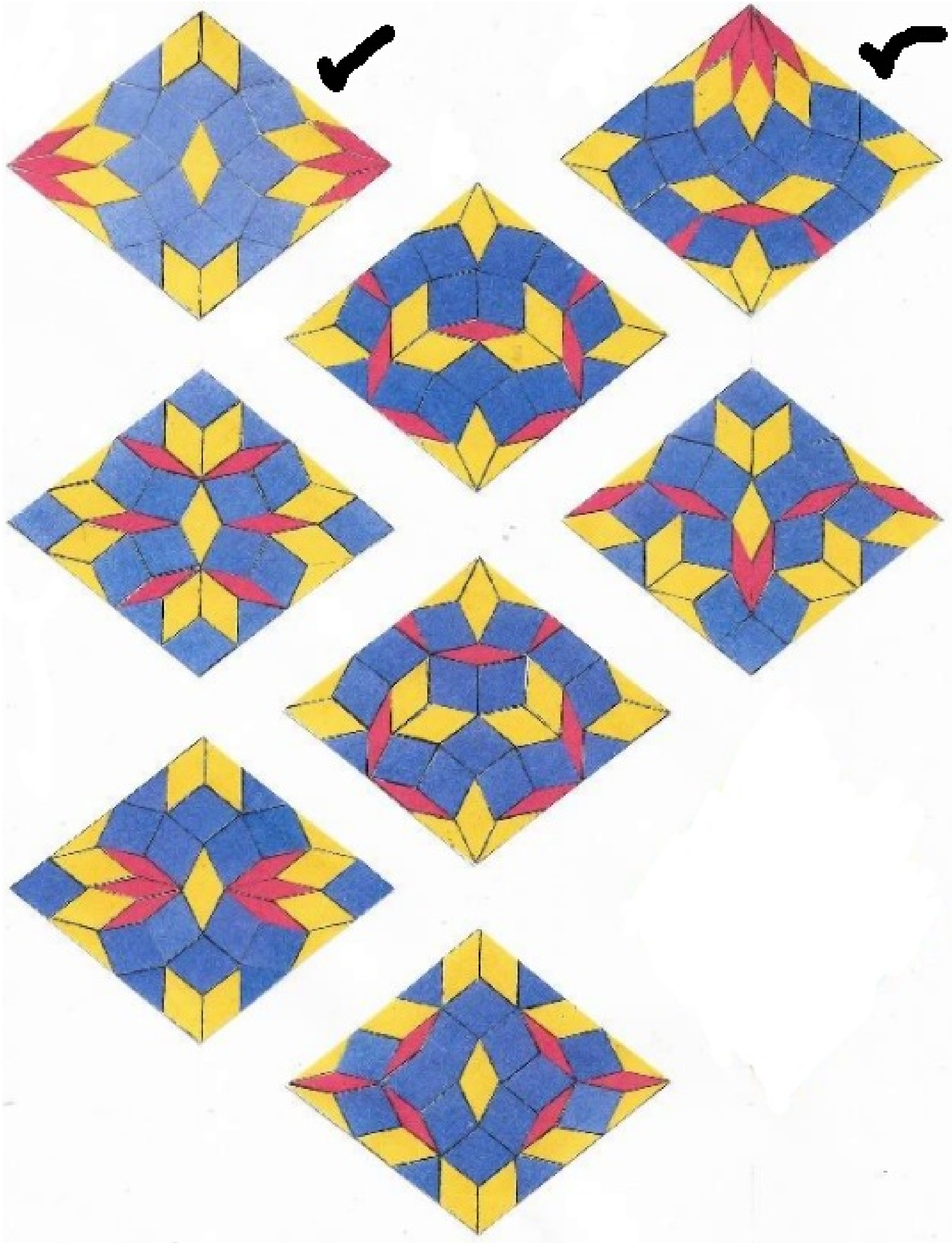
$D_B = 1 + 2\, d_A + 4\, D_B$
$\quad = 2 + d_A + 3\, D_B + d_C$
$\quad = 1 + 2\, D_B + 2\, d_C$



Substitutions for Type C Rhombi

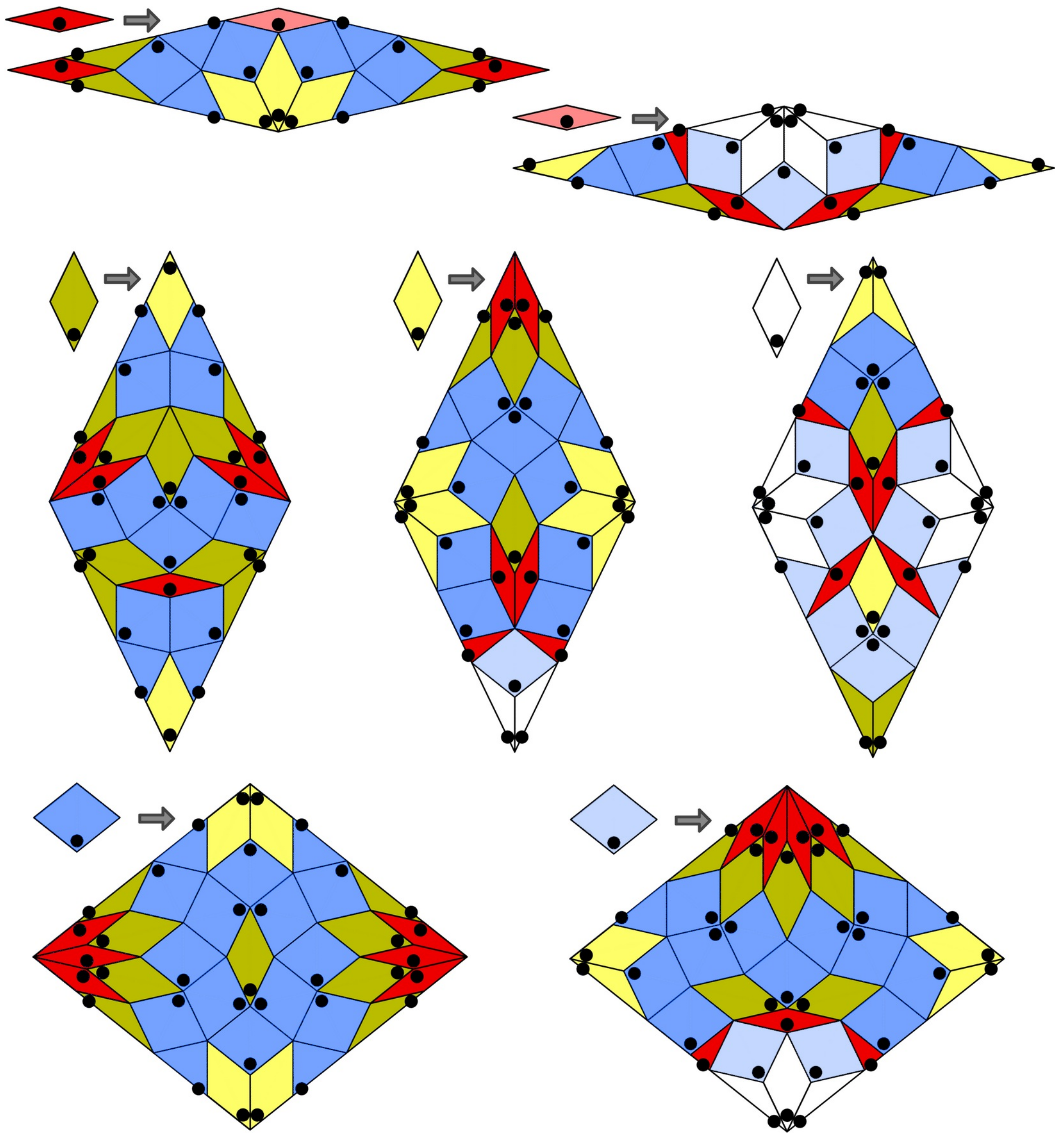
$\mathbf{C} = 6\,A + 11\,B + 14\,C$

$$\begin{aligned} d_C &= 2 + D_B + 2\,d_C \\ &= 1 + d_A + 2\,D_B + d_C \\ &= 2\,d_A + 3\,D_B \end{aligned}$$



Theo tried to use as few substitutions as possible...

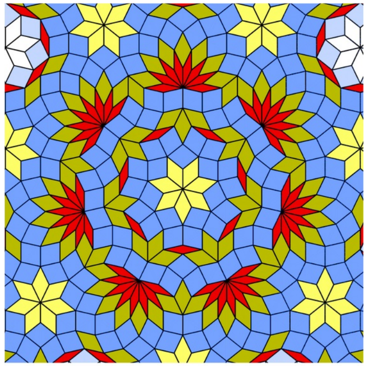
Substitution Rules for Theo's Tiling

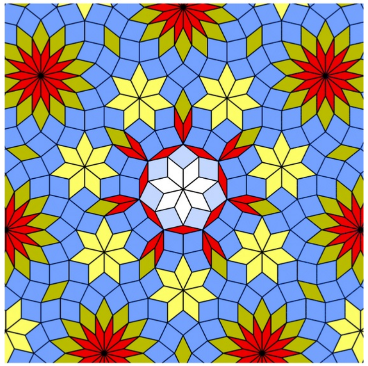


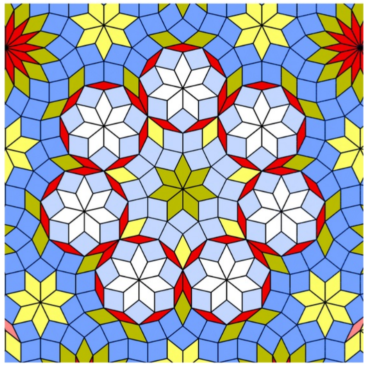
Correct orientations are important to get matching tiles after several inflations and substitutions.

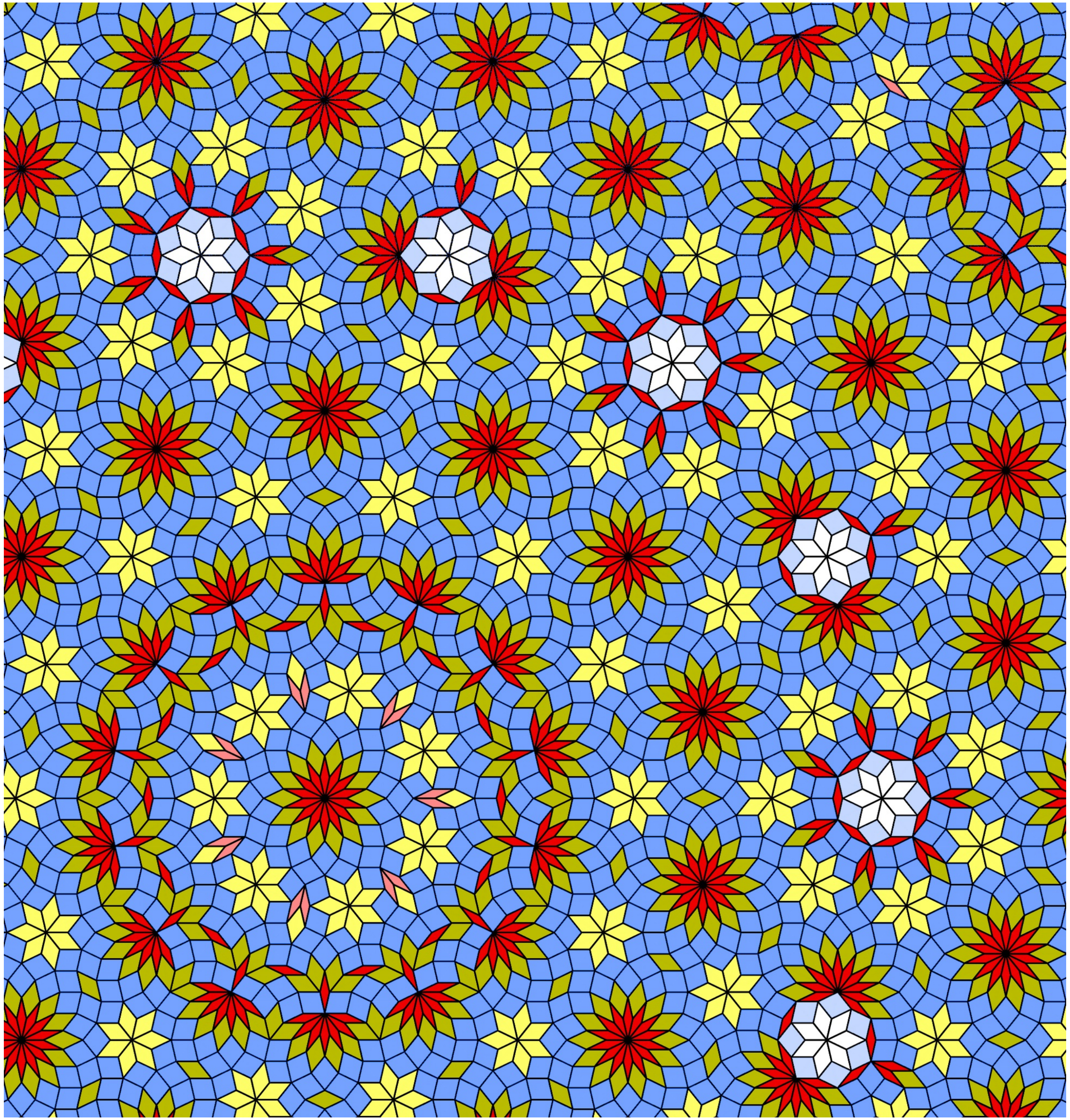
A small change in these rules gives a tiling of 14-fold rotational symmetry without mirror symmetry.

The Tiling









Conclusions

Ptolemy's theorem determines the products of diagonals of regular polygons, pentagon, heptagon, nonagon, ...

Rhombic rosettes are related to quasiperiodic tilings and define its rhombi.
For five-fold rotational symmetry we have two different types of rhombi, seven-fold has three types, ...

This defines:

- The possible inflation ratios for quasiperiodic tilings.
- The number of small rhombi substituting an inflated rhombus.
- The substitution along borders and diagonals of inflated tiles.

We have shown examples for five- and sevenfold rotational symmetry.

This method can be used for other rotational symmetries, except for orders that are multiples of 4.

Images of tilings have been generated with
http://geometricolor.ch/qpg/quasiperiodicGenerator/quasiperiodicAndFractal.html
using *.json text files.

You can download this presentation at
http://geometricolor.ch/images/geometricolor/presentation.pdf

Number of Rhombi in the Penrose Rhombus Tiling

Repeating inflation and substitution: Number of rhombi?

$n_A(i)$ and $n_B(i)$ are numbers of A and B rhombi after the i-th substitution.

From:
A = A + B
B = A + 2B

Follows the recursion:
$n_A(i+1) = n_A(i) + n_B(i)$
$n_B(i+1) = n_A(i) + 2\, n_B(i) = n_B(i) + n_A(i+1)$

Beginning with one B rhombus and growing the tiling we get these (n_A,n_B) pairs:

(0,1), (1,2), (3,5), (8,13), ...

Which are the Fibonacci numbers, because of the recursion relation above.

Large Fibonacci numbers are proportional to powers of the golden ratio τ.

Thus in the Penrose rhombus tiling the number of B rhombi is larger than the number of A rhombi by the golden ratio τ.